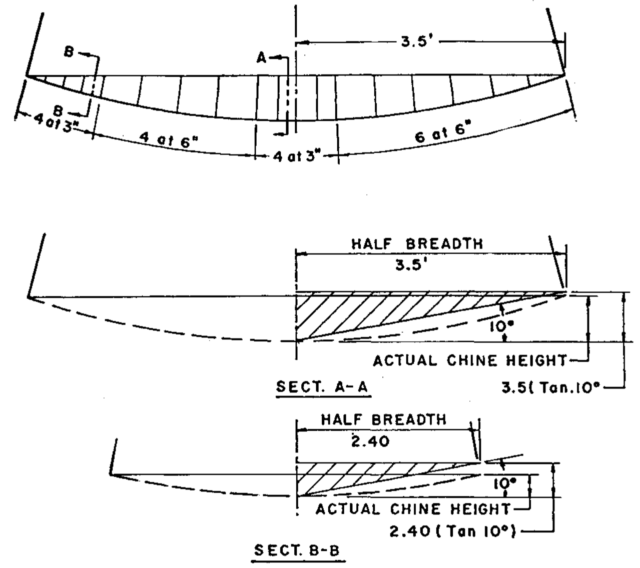

Fig. 1 Wedge configuration of the hull bottom. A 10° deadrise angle is assumed for each wedge, and the corresponding chine height is adjusted to comply with the half-breadth

are adjusted using the actual beam of the capsule. This retains the circular section in planform. The thickness of the wedges is reduced in the areas near the points of initial impact to decrease the effect of immersion of an individual wedge.

It will be noted, when considering Figs. 2 and 3 that, although the peak values have been attained satisfactorily, the acceleration onset rate lags the experimental results. This characteristic is inherent in the prediction, since a fixed deadrise angle, independent of immersion depth, is used. The fixed deadrise angle (10°) should be employed as a lower limit, since the theoretical results, using the Wagner coefficient for wedge impacts, are uncertain for smaller angles.

The steps in the curves are caused by the abrupt immersion of a finite wedge element (rise) and the abrupt immersion of a chine (drop). This effect can be reduced by a finer subdivision of wedges.

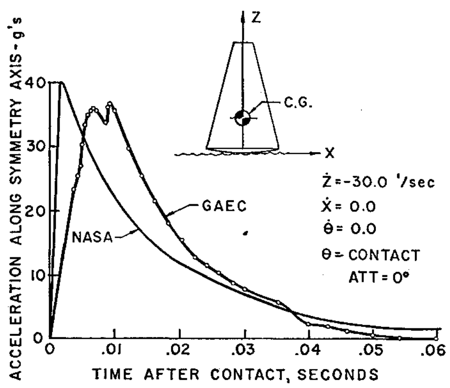

Fig. 2 Correlation for contact attitude of 0°

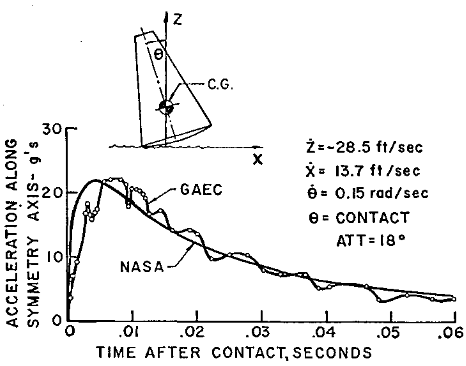

Fig. 3 Correlation for contact attitude of 18°

It must be noted that the method should be used primarily as a first approximation, basically for design purposes. When used in this way, the value of the procedure is evident. Any advances in the technique to increase accuracy must be accompanied by considerable theoretical advances in the field of either wedge immersion or immersion of other shapes.

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Class of Exact Solutions of Nonisentropic, One-Dimensional Magnetohydrodynamic Flow

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With a suitable choice of dependent variables, problems in one-dimensional unsteady gasdynamics may be reduced to finding solutions of a particular Monge-Ampère partial differential equation.[1, 2] If the particle paths and isobars coincide, this technique fails, so that an alternative procedure must be employed. This particular class of flows was the subject of a recent paper by Weir,[3] and it is the purpose of the present paper to show that Weir's discussion may be extended to magnetohydrodynamic flows subjected to a transverse magnetic field and that the resultant solutions reduce, in the limit of vanishing magnetic field, to those obtained by Weir.

The one-dimensional unsteady motion of an ideal, inviscid, perfectly conducting, compressible fluid subjected to a transverse magnetic field, i.e., the induction $\mathbf{B} = (0,0,B)$, is governed by the system of equations[4]

$$\rho_t + \rho u_x + \rho_x u = 0 \tag{1}$$

$$\rho(u_t + uu_x) + P_x + BB_x/\mu = 0 \tag{2}$$

$$B_t + uB_x + Bu_x = 0 \tag{3}$$

$$s_t + us_x = 0 \tag{4}$$

$$P = \exp[(s - s^*)/c_v]\rho^\gamma \tag{5}$$

where $\rho,u,P,s,s^*,\mu,b^2 = B^2/\mu\rho$, and γ are, respectively, the density, particle velocity, pressure, specific entropy, specific entropy at some reference state, permeability, square of the Alfvén speed, and ratio of specific heat at constant pressure c_p and at constant volume c_v. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone.

The characteristics of this system are given by

$$dx/dt = u,\ u + \omega,\ u - \omega \tag{6}$$

where $\omega = [b^2 + c^2]^{1/2}$, the true speed of sound, is the limiting case of a fast wave, and c is the local speed of sound. Further, $\rho[dx - u\,dt]$ is the exact differential of a function ψ,

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such that ψ equated to a constant defines the particle paths. From its definition, ψ satisfies the equation

$$\psi_t + u\psi_x = 0 \tag{7}$$

If it is assumed that $P = P(\psi)$, it follows from Eq. (7) that

$$P_t + uP_x = 0 \tag{8}$$

Since P, ρ, and s are related by some equation of state, Eqs. (4) and (8) yield

$$\rho_t + u\rho_x = 0 \tag{9}$$

so that Eq. (1) reduces to

$$\rho u_x = 0 \tag{10}$$

and $u = u(t)$ alone, say

$$u = -f'(t) \tag{11}$$

From Eqs. (3) and (10), B satisfies the equation

$$B_t + uB_x = 0 \tag{12}$$

Thus it is seen that B,P,ρ,s, and ψ all satisfy the same partial differential equation. Further, no explicit assumption of an equation of state has been made in the derivation to this point, and this class of flows may be generated from the single assumption $B = B(\psi)$.

By substituting Eq. (11) into Eqs. (4, 7–9, and 12), it is seen that s,ψ,P,ρ, and B are all functions of $y \equiv x + f(t)$. Equation (2) may be written as

$$-\rho f''(t) + (dP/dy) + (B/\mu)(dB/dy) = 0 \tag{13}$$

Since P and B are constant along a trajectory, Eq. (13) reduces to $f''(t) = 2A$, where A is constant, and this gives $f(t) = At^2 + A_1t + A_2$. With no loss of generality, A_2 may be taken as zero, and, if $u(0) = u_0$, it follows that $u(t) = -2At + u_0$, and the trajectories are given by

$$y = x + f(t) = x + At^2 - u_0t = \text{const} \tag{14}$$

i.e., a family of coaxial parabolas.

From the definition of ψ, $d\psi = \rho dy$, so that from Eq. (2)

$$\rho = (1/2A)(d/dy)[P + (B^2/2\mu)] \tag{15}$$

and since ρ is constant along a trajectory, Eq. (15) may be integrated to give

$$P + B^2/2\mu = 2A\psi + a$$

with some constant a. Thus, there is the result.

There exists only one possible case in which $P = P(\psi)$, or, equivalently, $B = B(\psi)$. Therein, $P + B^2/2\mu$ is a linear function of ψ; u is a linear function of t; and the trajectories are parabolas.

For a polytropic gas [Eq. (5)],

$$\begin{aligned} c^2 &= \frac{\gamma P}{\rho} = \frac{2A\gamma P}{(d/dy)[P + (B^2/2\mu)]} \\ \omega^2 &= \frac{B^2}{\mu\rho} + \frac{2A\gamma P}{(d/dy)[P + (B^2/2\mu)]} \end{aligned} \tag{16}$$

For the particular class of flows under consideration, the characteristics [Eq. (6)] are given by the trajectories and the curves $dx/dt = u \pm \omega$, i.e.,

$$dy/dt = \pm\omega \tag{17}$$

with the solutions

$$t = \pm \int \left[\frac{d[P + B^2/2\mu]/dy}{b^2d[P + B^2/2\mu]/dy + 2A\gamma P}\right]^{1/2} dy \tag{18}$$

If the equation of state is prescribed, e.g., Eq. (5), and also one initial condition, e.g., $s = s(\psi)$ or $P = P(x)$ at $t = 0$, the solution may be determined uniquely.

Example 1: Isentropic Flow with B = Constant

If Eq. (5) is written as $P = K\rho^\gamma$ with a constant K, it follows from Eq. (15) that

$$2Ay = \int [K/P]^{1/\gamma}\, dP$$

so that

$$P = [2A(\gamma - 1)y/\gamma K^{1/\gamma}]^{\gamma/(\gamma-1)} \tag{19}$$

From Eqs. (15) and (19),

$$\rho = [2A(\gamma - 1)y/\gamma K]^{1/(\gamma-1)} \tag{20}$$

Thus, $c^2 = 2Ay$ and $\omega^2 = 2Ay + B^2/\mu\rho$, where ρ is given by Eq. (20). Thus, if it is assumed that $u_0 = 0$ and $u = -2At$, then $c^2 = 2A(x + At^2)$.

Example 2

Assume that the initial condition $(P + B^2/2\mu) = (P_0 + B_0{}^2/2\mu)\exp(mx)$, with P_0, B_0, and m constants. Then it follows that $(P + B^2/2\mu) = (P_0 + B_0{}^2/2\mu)\exp(my)$ throughout the flow, and, further,

$$\begin{aligned} \rho &= \{m[P_0 + B_0{}^2/2\mu]\exp(my)\}/2A \\ c^2 &= 2A\gamma P_0/m[P_0 + B_0{}^2/2\mu] \\ \omega^2 &= 2A(\gamma P_0 + B_0{}^2/\mu)/m(P_0 + B_0{}^2/2\mu) \end{aligned}$$

Thus, the speed of propagation of small disturbances is constant. If it is assumed that $u_0 = 0$, then $u = -2At$, and the trajectories are given by $x + At^2 = \text{const}$. The Γ_+ and Γ_- characteristics are given by

$$x + At^2 = \pm\{[2AB_0{}^2/\mu + 2A\gamma P_0]/[m(P_0 + B_0{}^2/2\mu)]\}^{1/2}t + \text{const}$$

Graphs of these characteristics are given in Weir's paper. Finally, the entropy distribution is given by $s = c_vm(1 - \gamma)y + \text{const}$. In this example, the entropy gradient causes the sound speed to remain constant.

Example 3

Assume that $P = \alpha y$, $B^2/2\mu = \beta y$. Then

$$\rho = (\alpha + \beta)/2A \qquad c^2 = 2A\gamma\alpha y/(\alpha + \beta)$$
$$\omega^2 = 2Ay(\gamma\alpha + 2\beta)/(\alpha + \beta)$$

so that the density remains constant. The characteristics are given by

$$t = \pm[2(\alpha + \beta)(x + At^2)/A(2\beta + \gamma\alpha)]^{1/2} + \lambda \tag{21}$$

where λ is a constant of integration; thus it is seen that the Γ_+ and Γ_- characteristics are given by the same family of parabolas, viz.,

$$A\alpha(\gamma - 2)t^2 - 2\lambda(2\beta + \gamma\alpha)At - 2(\alpha + \beta)x + A(2\beta + \gamma\alpha)\lambda^2 = 0 \tag{22}$$

The envelope of Eq. (22) is $x + At^2 = 0$, so that the conclusion in this case agrees with that obtained by Weir, viz., Eq. (21) implies that the point at which any member of the family given by Eq. (22) touches the envelope is also the point that divides that part of a curve representing a Γ_+ characteristic from that representing a Γ_- characteristic. Since this example deals with a gas expanding into a vacuum, and $x + At^2 = 0$ is the particle path through the origin, it must be the front of the expanding gas. Thus, this gives another example where a Γ_- characteristic meets a gas front at a tangent and is reflected off as a Γ_+ characteristic. This occurs because the slopes of the Γ_+ and Γ_- characteristics are the same at the front, since $\omega = 0$ there. A sketch of the characteristics is given in Weir's paper. Finally, the entropy distribution is given by $s = c_v \log y + \text{const.}$

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Final-Stage Decay of a Single Line Vortex

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I. Vortex Generation and Decay

The generation of a circulatory flow field by a vortex core of radius r_g was studied by Goldstein.[1] The differential system governing the fluid motion is

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right)$$
$$t < 0: \quad v \equiv 0 \tag{1}$$
$$t > 0: \quad r = r_g, v = v_g; \; r = \infty, v = 0$$

and has the solution

$$v = \frac{v_g r_g}{r} + \frac{2v_g}{\pi}\int_0^\infty \exp(-\nu t x^2)\,\frac{J_1(xr)Y_1(xr_g) - Y_1(xr)J_1(xr_g)}{J_1^2(xr_g) + Y_1^2(xr_g)}\,\frac{dx}{x} \tag{2}$$

The limiting case of a vanishing core radius but finite centerline circulation Γ_g was shown by Rouse and Hsu[2] to reduce to following form:

$$v = (\Gamma_g/2\pi r)\exp(-r^2/4\nu t) \tag{3}$$

This corresponds to the vorticity diffusion from a line vortex of strength Γ_g, i.e.,[3]

$$\zeta = (\Gamma_g/4\pi\nu t)\exp(-r^2/4\nu t) \tag{4}$$

Thus the generation of a circulatory flow field by a vortex core is through vorticity diffusion, and, with finite time of generation, the velocity field so generated has limited extent. Moreover, such a velocity field is never and nowhere free of vorticity. Indeed, a potential flow never can be generated by a shear mechanism alone in finite time; it only represents a limit solution for $t \to \infty$. Thus, as $t \to \infty$, Eq. (3) describes a potential flow.

The spatial growth of velocity field and the time decay of kinetic energy of a viscous vortex after a generation period t_g can be studied simply by introducing, at $t = t_g$, an anti-circulation along the centerline. By simple superposition, one readily can write down the following result:

$$t = t_g + t_d$$
$$v = \frac{\Gamma_g}{2\pi r}\left[\exp\frac{-r^2}{4\nu(t_g + t_d)} - \exp\frac{-r^2}{4\nu t_d}\right] \tag{5}$$

This is indeed nothing else but the solution to the well-known Rayleigh "start-then-stop" problem for a circulatory flow field. The catching-up process between these two outward-propagating, oppositely orientated velocity fields results in a gradual annihilation of the momentum and a continuous decay of the kinetic energy of the flow field.

Equation (5) describes the velocity field of a viscous vortex at time t_d after its generation. It is clear that this flow field has not only limited spatial extent but also finite energy content.

Writing $A = \Gamma_g$, $a = 4\nu t_d$, $k = 1 + t_g/t_d$, Eq. (5) can be put in the form

$$v = (A/2\pi r)[\exp(-r^2/ak) - \exp(-r^2/a)] \tag{6}$$

It is easy to see that constants A, a, k can be used to define an initial vortex field, provided that k is not taken too close to unity. The parameter k indeed fixes the age of the initial vortex field; for $t_d/t_g \to 0$, $k \to \infty$, and for $t_d/t_g \to \infty$, $k \to 1$.

A, a, k can be related further to the measurable field characteristics r_0, Γ_0, ζ_0 as follows: defining r_0 as the initial radius of maximum circulation Γ_0 ($\Gamma = 2\pi r v$), i.e., for

$$\frac{\partial \Gamma}{\partial r} = \frac{2Ar}{a}\left(\exp\frac{-r^2}{a} - \frac{1}{k}\exp\frac{-r^2}{ka}\right) = 0$$

one has

$$r_0 = \{[ka/(k-1)]\ln k\}^{1/2} \tag{7}$$
$$\Gamma_0 = A(k-1)k^{k/(1-k)} \tag{8}$$

Now, vorticity is given by

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(vr) = \frac{A}{\pi k a}\left(k\exp\frac{-r^2}{a} - \exp\frac{-r^2}{ka}\right)$$

so, putting $r = 0$, one has for the centerline vorticity

$$\zeta_0 = (A/\pi k a)(k-1) \tag{9}$$

For a particular set of A, a, k (or initial values r_0,ζ_0,Γ_0), the subsequent velocity field now can be put in the form

$$v = \frac{A}{2\pi r}\left(\exp\frac{-r^2}{ak + 4\nu t} - \exp\frac{-r^2}{a + 4\nu t}\right) \tag{10}$$

The vortex-core radius r_c ($r = r_c$, $\partial\Gamma/\partial r = 0$) readily can be obtained as follows:[2]

$$r_c^2 = a\frac{(k + 4\nu t/a)(1 + 4\nu t/a)}{k - 1}\ln\frac{k + 4\nu t/a}{1 + 4\nu t/a} \tag{11}$$

The kinetic energy per unit length of vortex is given by an integral:

$$E = \pi\rho\int_0^\infty v^2 r\,dr = \frac{\rho A^2}{8\pi}\int_0^\infty \frac{1}{r^2}\left(\exp\frac{-r^2}{ka + 4\nu t} - \exp\frac{-r^2}{a + 4\nu t}\right)^2 2r\,dr$$

Upon introducing the identity

$$\int_0^\infty [\exp(-px) - \exp(-qx)]^2\frac{dx}{x} = \ln\frac{p+q}{2p} + \ln\frac{p+q}{2q}$$

one obtains[2]

$$E = \frac{\rho A^2}{8\pi}\ln\frac{[1 + (k + 4\nu t/a)/(1 + 4\nu t/a)]^2}{4(k + 4\nu t/a)/(1 + 4\nu t/a)} \tag{12}$$

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